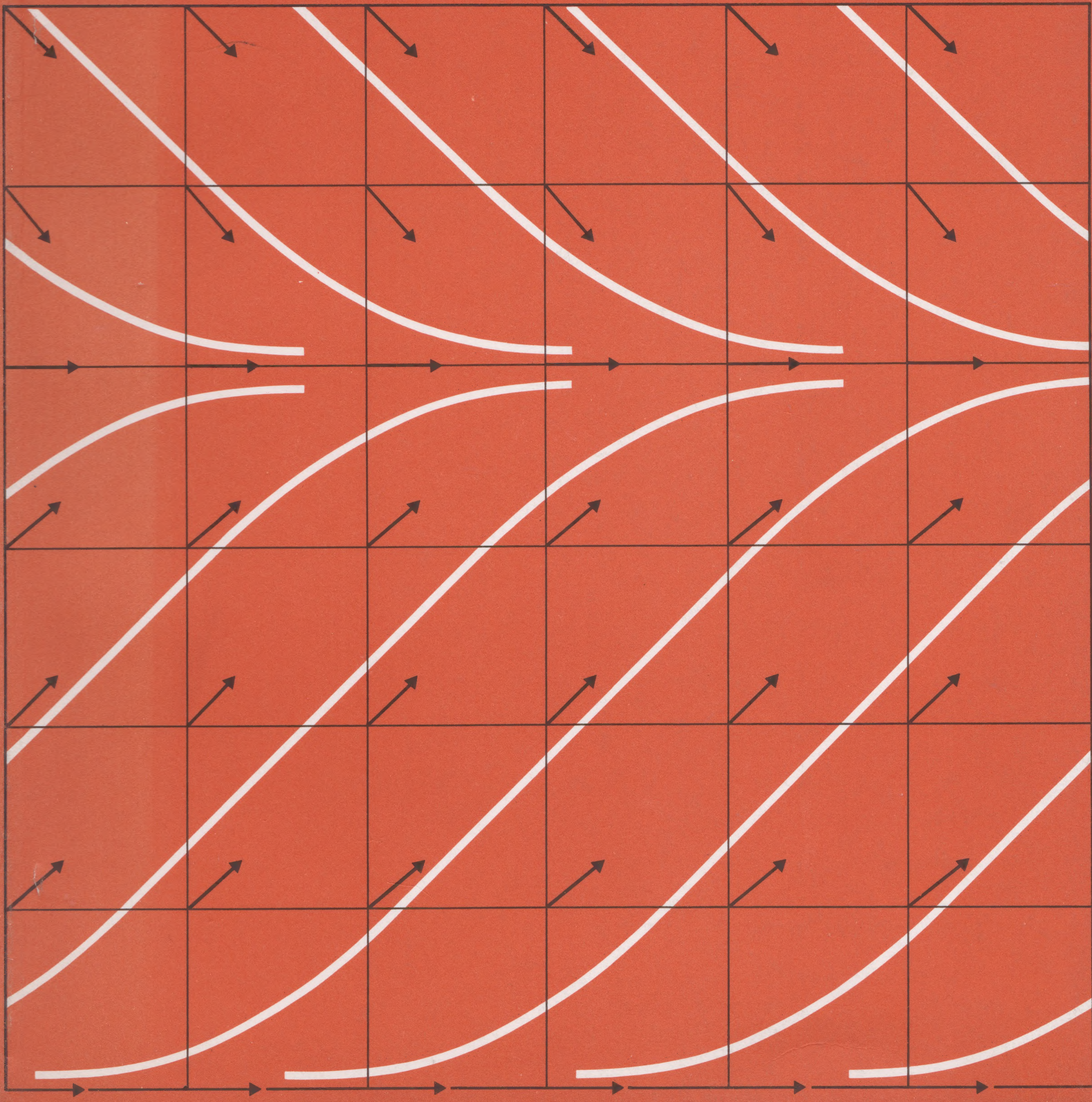
M100 24

THE OPEN UNIVERSITY

Mathematics Foundation Course Unit 24

Differential Equations I



The Open University

Mathematics Foundation Course Unit 24

DIFFERENTIAL EQUATIONS I

Prepared by the Mathematics Foundation Course Team

Correspondence Text 24

The Open University Press

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Contents

Objectives

The principal objective of this unit is to introduce differential equations of the first order and to discuss their solutions and some methods of solving them.

After working through this unit you should be able to:

(i) explain the meanings of the following terms:
  solution set of a differential equation,
  family of solution curves,
  separation of variables method,
  integrating factor method,
  Euler's method of solving first order differential equations;

(ii) sketch a family of solution curves for a given simple first order differential equation;

(iii) solve a given first order differential equation by either the separation of variables method or the integrating factor method (in appropriate cases);

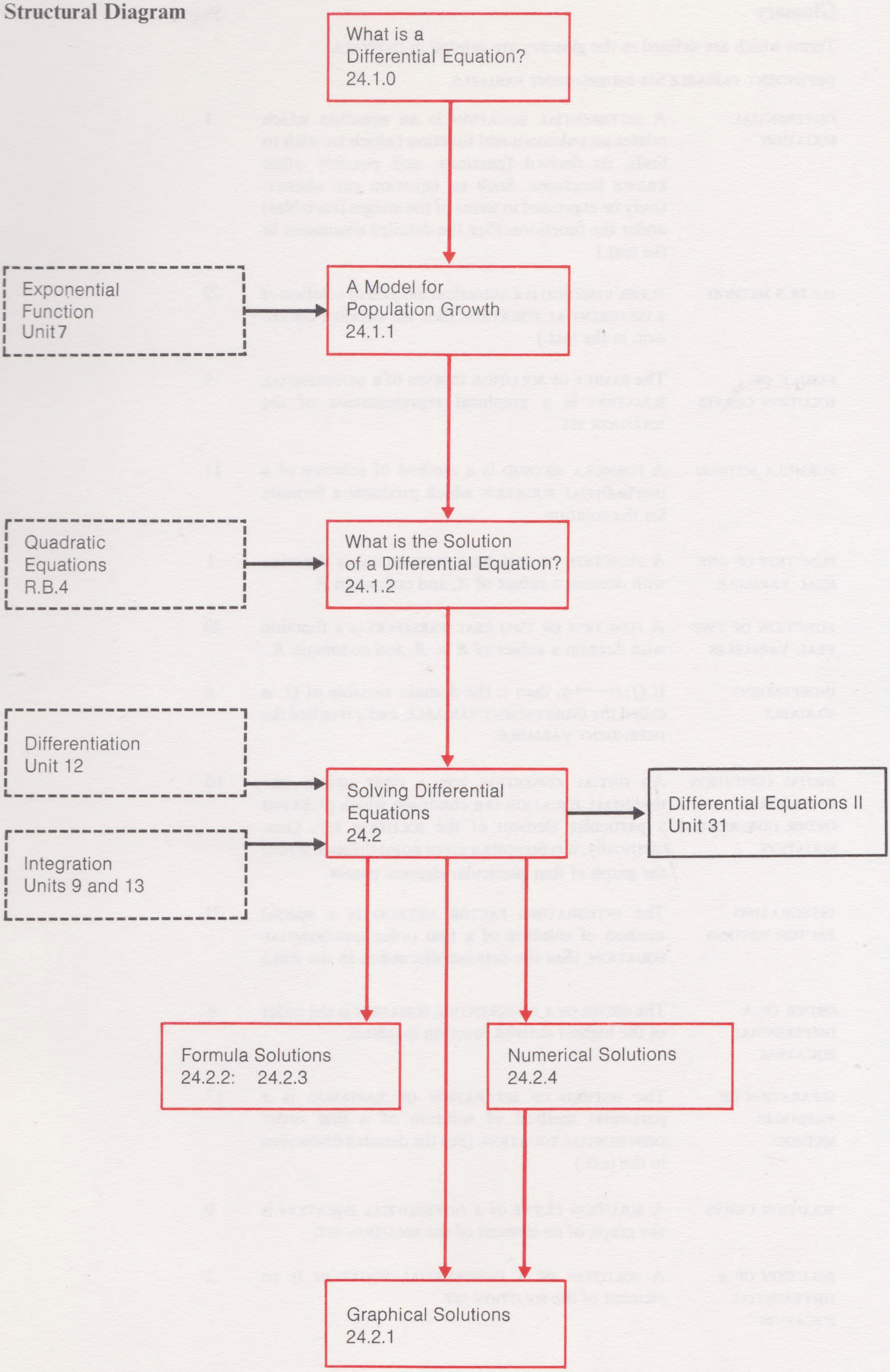
(iv) having completed (iii), find a particular solution with a given initial condition;

(v) apply Euler's method, given a first order differential equation, an initial condition and a specified step length.

Note

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



Glossary

Terms which are defined in the glossary are printed in CAPITALS.

DEPENDENT VARIABLE See INDEPENDENT VARIABLE.

		Page
DIFFERENTIAL EQUATION	A DIFFERENTIAL EQUATION is an equation which relates an unknown real function (which we wish to find), its derived functions, and possibly other known functions. Such an equation can alternatively be expressed in terms of the images (variables) under the functions. (See the detailed discussion in the text.)	1
EULER'S METHOD	EULER'S METHOD is a numerical method of solution of a DIFFERENTIAL EQUATION. (See the detailed discussion in the text.)	29
FAMILY OF SOLUTION CURVES	The FAMILY OF SOLUTION CURVES of a DIFFERENTIAL EQUATION is a graphical representation of the SOLUTION SET.	9
FORMULA METHOD	A FORMULA METHOD is a method of solution of a DIFFERENTIAL EQUATION which produces a formula for the solution.	11
FUNCTION OF ONE REAL VARIABLE	A FUNCTION OF ONE REAL VARIABLE is a function with domain a subset of R, and codomain R.	1
FUNCTION OF TWO REAL VARIABLES	A FUNCTION OF TWO REAL VARIABLES is a function with domain a subset of $R \times R$, and codomain R.	29
INDEPENDENT VARIABLE	If $Q: t \longmapsto q$, then t, the domain variable of Q, is called the INDEPENDENT VARIABLE, and q is called the DEPENDENT VARIABLE.	6
INITIAL CONDITION FOR A FIRST ORDER DIFFERENTIAL EQUATION	An INITIAL CONDITION FOR A FIRST ORDER DIFFERENTIAL EQUATION is a condition which picks out a particular element of the SOLUTION SET. Geometrically, it represents a given point through which the graph of that particular element passes.	10
INTEGRATING FACTOR METHOD	The INTEGRATING FACTOR METHOD is a special method of solution of a first order DIFFERENTIAL EQUATION. (See the detailed discussion in the text.)	21
ORDER OF A DIFFERENTIAL EQUATION	The ORDER OF A DIFFERENTIAL EQUATION is the order of the highest derived function involved.	6
SEPARATION OF VARIABLES METHOD	The METHOD OF SEPARATION OF VARIABLES is a particular method of solution of a first order DIFFERENTIAL EQUATION. (See the detailed discussion in the text.)	17
SOLUTION CURVE	A SOLUTION CURVE of a DIFFERENTIAL EQUATION is the graph of an element of the SOLUTION SET.	9
SOLUTION OF A DIFFERENTIAL EQUATION	A SOLUTION OF A DIFFERENTIAL EQUATION is an element of the SOLUTION SET.	2

		Page
SOLUTION SET OF A DIFFERENTIAL EQUATION	The SOLUTION SET OF A DIFFERENTIAL EQUATION is the set of all real functions which satisfy the DIFFERENTIAL EQUATION.	2
STEP LENGTH	The STEP LENGTH is the width of interval used to determine the next point on an estimated SOLUTION CURVE of a first order DIFFERENTIAL EQUATION.	30

Bibliography

Most textbooks go well beyond the scope of this text. We have therefore chosen just two to give you some idea of the number of classical methods available, and to give a different viewpoint of some of the material presented in this text. They are

H. Betz, P. B. Burcham and G. M. Ewing, *Differential Equations with Applications* (Harper and Row, 1964).
Parts of Chapters 1–4 are relevant.

W. Kaplan, *Elements of Ordinary Differential Equations* (Addison-Wesley, 1964).
Parts of Chapters 1–3 are relevant.

For the numerical approach, we suggest:

N. Macon, *Numerical Analysis* (John Wiley, 1963).
Chapter 10 gives an indication of the more formal approach to the numerical methods and their accuracy. You will probably find this last reference rather hard at this stage.

General Note

In this unit we shall use, as appropriate and convenient, any of the calculus notations mentioned previously in the course. For instance, if Q is a real function, such that

$$Q: t \longmapsto Q(t) \qquad (t \in R)$$

and we write $q = Q(t)$, then we write the derived function as Q', when the "function" notation suits us, or we express the derivative as

$$Q'(t) = \frac{dq}{dt},$$

using the "function image" notation or the Leibniz notation. You should by now be reasonably familiar with the function notation as we have used it throughout the course, and also with the distinction between the function Q and the variables t and q, where t is a general element of the domain of Q, and q is a general element of the image set. Our objective in using these various notations is to enable you not only to read this course, but also to read any other texts that you may come across now and in further studies.

24.1

24.1.0

Introduction
* * *

24.1 INTRODUCTION

24.1.0 What is a Differential Equation?

In *Unit 12, Differentiation I* we saw that the derivative of a real function Q represents the rate of change of the image value under Q, or the slope of the (tangent to the) pictorial graph of Q; that is, $Q'(t)$, $(t \in R)$, is the rate of change of $Q(t)$ at t. Given the function Q, we can derive Q' by using one of the standard rules. For example, if

$$Q : t \longmapsto e^t - 2t^3 \qquad (t \in R),$$

then

$$Q' : t \longmapsto e^t - 6t^2 \qquad (t \in R).$$

This is straightforward.

The problem is appreciably harder when we are given the derived function Q' and are asked to determine Q. This is the problem of *integration* and we have developed some tools to cope with this (*Unit 9* and *Unit 13, Integration I and II*). Essentially, there are two main points. In the first place, the differentiation operator D is a function, but the reverse of D is not a function. We coped with this by introducing a constant function and obtained the images of Q' under the reverse mapping in the form

any primitive + constant function.

For example, if we are given

$$Q' : t \longmapsto 2t \qquad (t \in R)$$

then

$$Q : t \longmapsto t^2 + c \qquad (t \in R),$$

where c is a real number, and for each choice of c we get an image of Q' under the integration mapping. The second important point is that very often we could not find a simple expression for the integral (or, even if we could, the labour involved was prohibitive).

For example, if

$$Q' : t \longmapsto \frac{1}{\sqrt{t^3 + 1}} \qquad (t \in R^+)$$

then

$$Q : t \longmapsto ?$$

In this unit we are going to make our integration process still more difficult; not because it is "good for the soul", but because of the wide field of application of the development. We are on the threshold of the subject known as *differential equations*, which has a considerable literature associated with it and on which much research is still undertaken today.

A *differential equation* of the type which we consider in this unit is an equation involving an unknown real function and its derived functions, and possibly other known functions. Thus if f is a real function,

$$f' + 2f = O$$

(where O is the function $x \longmapsto 0$ $(x \in R)$) is a differential equation, as is

$$f' + (x \longmapsto x^2) \times f = x \longmapsto \sin x.$$

The simplest form of differential equation is an equation of the form

$$f' = g$$

where g is a known real function; its solution presents us with precisely the integration problem we have discussed previously. We shall discuss more

fully in section 24.1.2 what we mean by the *solution* of a differential equation, but we note here that the solution of such an equation is a *function* (or *set of functions*). This contrasts with most previous equations we have solved in this course, where in each case the solution has usually been a *number* (or a *set of numbers*). In general, in mathematics an equation is a relation defined on a set of elements and can be considered as defining a subset consisting of those elements which belong to the solution set of the relation. (For the definition of *relation*, see *Unit 19*.) In the language of section 17.2.1, *Unit 17*, *Logic II*, an equation is an open sentence, which becomes a true proposition when an element from the solution set is substituted for the variable.

Thus a differential equation is a relation defined on a set of (real) functions, and so its solution set is a subset consisting of functions. For example, one element of the solution set of the differential equation

$$f' + 2f = O$$

is the function

$$f : x \longmapsto -2\exp(-2x) \qquad (x \in R).$$

Note that a relation between functions can be written in terms of the images under the functions. For example,

$$f' + 2f = O$$

is equivalent to

$$f'(x) + 2f(x) = 0,$$

for all x in the domain of f'. We shall refer to the "image form" of a differential equation as a differential equation too; in this case it is frequently convenient to use the Leibniz notation for the derivative.

Differential equations arise directly from many basic physical laws and are therefore fundamental to the study of considerable parts of science and engineering. For example, one of Newton's laws states that

$$\textit{force applied} = \textit{mass} \times \textit{acceleration}.$$

Acceleration is the rate of change of velocity, and velocity depends on time. If, for instance, the force applied itself depends on the velocity, as it does in the car engine, then we can write the above equation in terms of a "position" function and its first and second derived functions, and hence obtain a differential equation. Differential equations do not arise solely from this kind of background: they also arise from such diverse sources as architecture, biology and economics. In this unit we are going to solve only a simple form of differential equation involving Q, Q' and known functions.

Our plan is to concentrate mainly on a few closely related differential equations. In section 24.1.1 we shall do the modelling; in other words, we shall develop some differential equations from plausible physical situations. In section 24.1.2 we shall discuss some basic ideas concerning the solution of differential equations. We shall then investigate graphical solutions of differential equations, since the pictorial approach will give us a good idea of how solutions behave. This approach will also enable us to appreciate the numerical approach to differential equations which is introduced in section 24.2.4. In sections 24.2.2 and 3 we introduce some which might be called *formula* methods of solution of differential equations. At this particular stage the objective of a formula method is to rearrange the differential equation in such a way that we can use our powerful tool—integration.

This is the approach to problem-solving which Polya stresses. We ask: "Is there a problem like it which we have already solved?" The answer may be "Yes" if we do a little rearranging first.

24.1.1 The Population Growth Problem

24.1.1

Discussion

In this section we take up a problem of population growth considered previously in section 7.4 of *Unit 7, Sequences and Limits I*, and look at the increasingly sophisticated mathematical models we can design.

Let $q = Q(t)$ represent (at any time $t \in R$) the population (in thousands) of a particular species, which we shall take to be human beings, but could equally well be viruses, locusts, fish or birds.

Since q is measured in thousands, it can really take only certain rational values corresponding to a whole number of human beings, e.g. 285.632. This brings us to our first modification in the process of formulating our mathematical model. To talk about growth, or rate of change, we want to talk about the *derived function*, and the derived function was certainly not defined for functions with this subset of the rationals as codomain since the limiting procedure we adopted in the definition then becomes meaningless. This looks like a full stop: but what we do is to take the codomain of Q to be R^+ in order to make our mathematical model. That is, we assume that the variable q can take *any* positive real values to enable us to use the powerful tools of calculus.

To determine how the population will change with time we need to introduce functions to represent the number of births and deaths per year. We shall call these B and M respectively. The number of births (deaths) usually depends on the size of the population, $Q(t)$, which itself depends on the time at which it is measured. Thus the composite function $B \circ Q$ will tell us how the number of births per year depends on the time.

The domain of each of the functions B and M is the subset of the reals which represents the population in thousands, and the codomain is a subset of the reals which represents the number of births (deaths) per year. We shall measure t in years. The rate of growth of population is then given by

$$DQ = B \circ Q - M \circ Q$$

or

$$Q'(t) = B(Q(t)) - M(Q(t)) = B(q) - M(q)$$

The simplest assumption to make is that the number of births and the number of deaths per year are both constant, i.e.

$$B : q \longmapsto b_0 \qquad (q \in R^+),$$

$$M : q \longmapsto m_0 \qquad (q \in R^+),$$

where b_0 and m_0 are known numbers.

Thus we get

$$Q'(t) = b_0 - m_0 = k_0, \text{ say.}$$

It is not our purpose to *solve* differential equations in this section, although this one has obvious solutions. All we are illustrating here is how differential equations arise. It is unrealistic to imagine that the number of births per year will remain constant; for example, if the population doubles it is likely that the number of births also doubles. To translate this into mathematical terms is to require that the function B be of the form

$$B : q \longmapsto b_1 q \qquad (q \in R^+),$$

where b_1 is a positive number. That is, the number of births is proportional to the number in the population. (The number b_1 is the birth rate in the usual official sense; that is, the number of births per thousand per year.) Similarly, it is reasonable to assume that the number of deaths is proportional to the population, i.e.

$$M : q \longmapsto m_1 q \qquad (q \in R^+),$$

where m_1 is a positive number. The differential equation is now

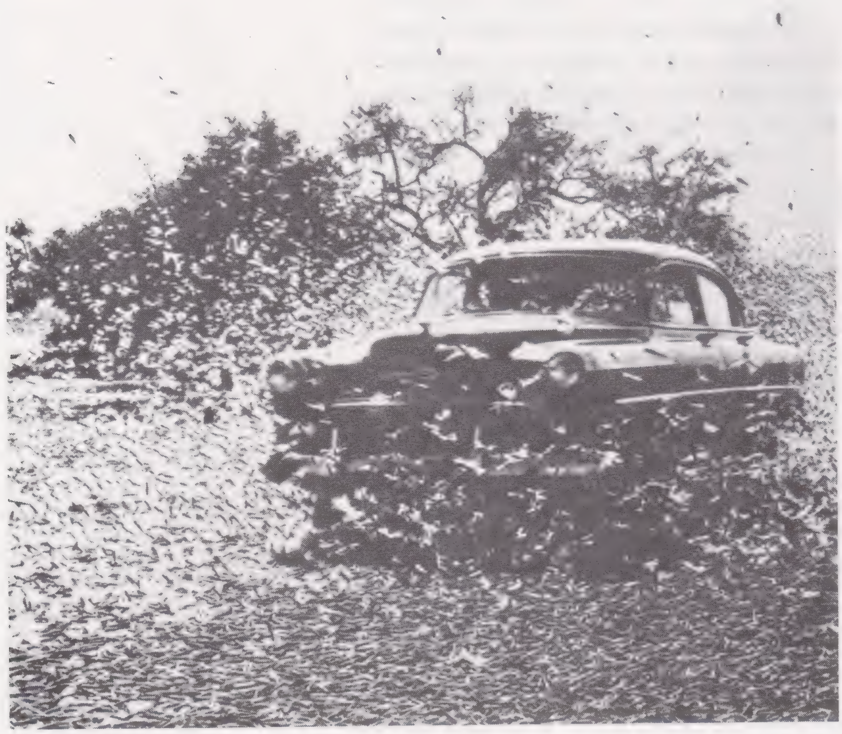
$$Q'(t) = b_1 Q(t) - m_1 Q(t) = k_1 Q(t)$$

or

$$\frac{dq}{dt} = k_1 q,$$

where $k_1 = b_1 - m_1$.

Now we introduce a further refinement to our model. As the population increases, so the available food supplies may become depleted. With a greater possibility of disease and disasters claiming a higher number of deaths, we would expect that the death rate would increase with increasing population.



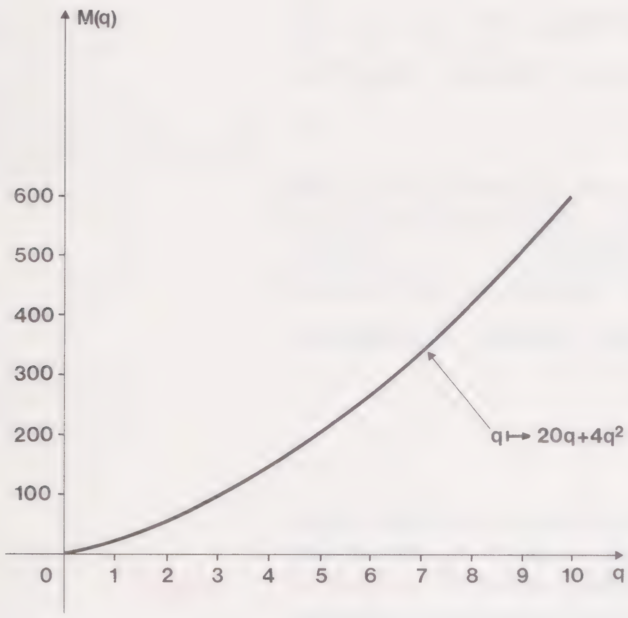
We could perhaps represent this mathematically by

$$M: q \longmapsto m_1 q + m_2 q^2 \qquad (q \in R^+)$$

where m_2 is a positive number. For instance, if we take $m_1 = 20$ and $m_2 = 4$, we would have

$$M: q \longmapsto 20q + 4q^2 \qquad (q \in R^+);$$

the graph of M is depicted in the figure.



This would imply that with a population of 1 000, i.e. $q = 1$, the death rate would be 24 per thousand per year, whereas if the same population were to increase to 10 000, i.e. $q = 10$, the death rate would increase to $\frac{200 + 400}{10} = 60$ per thousand per year. The differential equation would now be

$$Q'(t) = k_1 Q(t) - m_2 (Q(t))^2$$

or

$$\frac{dq}{dt} = k_1 q - m_2 q^2,$$

where $k_1 = b_1 - m_1$.

From now on in this text we shall write the "image form" of a differential equation in terms of either $Q(t)$ or q, rather than continue to write both versions. The first form, explicitly in terms of the images, is useful, since it tells us that we are looking for a *function* as a solution rather than a number. The second form, using the Leibniz form of the derivative, is convenient because it is more concise. We shall use either form depending on which we feel is more appropriate in the context.

It is interesting to notice that we can obtain some information from a differential equation even without solving it. We can write our differential equation in the form

$$\frac{dq}{dt} = m_2 q \left(\frac{k_1}{m_2} - q \right)$$

Assuming $k_1 > 0$ $\left(\text{if it were not, } \frac{dq}{dt} \text{ would be negative and the population would decrease until the model became inapplicable or the population became zero}\right)$ we see that, provided

$$q < \frac{k_1}{m_2},$$

$\frac{dq}{dt}$ is positive, implying increasing population. (Remember that $q = Q(t)$, i.e. q is a *number*.) If $q = \frac{k_1}{m_2}$, then the population is static, and if $q > \frac{k_1}{m_2}$, the derivative is negative and the population is decreasing. This tells us that $\frac{k_1}{m_2}$ is the only stable population.

This leads us to the final refinement we propose to make here. This stable population which can be supported is likely to increase with time, due to improvements in medicine, agricultural technology, etc. Let us suppose that this stable population increases linearly with time and that we replace $\frac{k_1}{m_2}$ by $(k_2 t + k_3)$, where k_2 and k_3 are positive numbers. The differential equation becomes

$$\frac{dq}{dt} = m_2 q\,(k_2 t + k_3 - q)$$

Notice that the right-hand side now involves the two variables q and t. These variables are customarily given special names; t, the domain variable of the function Q we wish to determine is called the independent variable; q is the dependent variable, since it is the variable in the codomain of the function Q we wish to determine.

Definition 1

Notice also that we have made various suppositions without any real justification. What we have is a very *tentative* model which, on analysis, may or may not fit the facts. In a real situation we would need to test its consequences against known data before we could attempt to make any predictions from it.

The various differential equations in this introduction are all differential equations of the *first order*. Order is determined by the highest derivative present. In other words, if a differential equation contained $\frac{d^n q}{dt^n}$ or $Q^n(t)$ and no higher derivatives, we would say it was nth *order* or *of order n*. For example, the equation

Definition 2

$$\left(\frac{d^3 q}{dt^3}\right)^2 = 9$$

is third order.

24.1.2 Basic Ideas about Solutions of Differential Equations

24.1.2

Discussion

* *

Consider the familiar problem of solving the quadratic equation

$$t^2 - 5t + 4 = 0 \qquad (t \in R).$$

The solution set of this equation, namely

$$\{t : t^2 - 5t + 4 = 0, t \in R\},$$

contains two members and is

$$\{4, 1\}.$$

Other solution sets in R for quadratic equations may contain two, one (in the case of two equal roots) or no members.

Exercise 1

Exercise 1
(2 minutes)

Write down examples of three quadratic equations whose solution sets contain two, one and no real members respectively. ■

Exercise 2

Exercise 2
(2 minutes)

Describe, in an explicit form, the members of the solution set of the equation

$$\sin t = 0 \qquad (t \in R)$$

Can you count how many members there are? ■

Discussion

* *

The solution set of a differential equation may be written in a similar way to that of an ordinary equation. Consider the differential equation

$$Q'(t) = k_0 \qquad (t \in R)$$

or

$$Q'(t) - k_0 = 0$$

The set of all functions which satisfy this differential equation can be denoted by

$$\{Q : Q'(t) - k_0 = 0 \qquad (t \in R)\}.$$

In this case each solution to the differential equation is simply a primitive function (indefinite integral) of $t \longmapsto k_0$, that is

$$t \longmapsto k_0 t + c \qquad (t \in R);$$

c is a constant of integration (see *Unit 13, Integration II*). Thus another (explicit) form of the solution set is

$$\{Q : Q = t \longmapsto k_0 t + c \qquad (t \in R), c \in R\},$$

which is often more concisely written in terms of images as

$$\{Q : Q(t) = k_0 t + c \qquad (t \in R), c \in R\}.$$

This set can also be written as a relation between variables:

$$\{Q : q = k_0 t + c \qquad (t \in R), c \in R\}.$$

We see again here the important difference between the solutions of ordinary equations defined on sets of numbers and the solutions of differential equations which are defined on sets of functions. In the former case the solution set has *real numbers* as members whilst in the latter case it has (*real*) *functions* as members. We can observe this visually by considering the graphical representation of the solution set of the equation

$$\sin t = 0 \qquad (t \in R),$$

(*continued on page 8*)

Solution 1

Solution 1

For example,

(i) $t^2 - 3t + 2 = 0$ whose solution set is $\{2, 1\}$.

(ii) $t^2 - 2t + 1 = 0$ whose solution set is $\{1\}$.

(iii) $t^2 - 2t + 2 = 0$ whose solution set has no real members. ■

Solution 2

Solution 2

The solution set can be written in the form

$$\{n\pi : n \in Z\},$$

since the sine of any integer multiple of π is zero. There are as many solutions as there are positive integers. The solutions can be put in one–one correspondence with the set of all positive integers as follows:

$$0 \longleftrightarrow 1$$

$$\pi \longleftrightarrow 2$$

$$-\pi \longleftrightarrow 3$$

$$2\pi \longleftrightarrow 4$$

$$-2\pi \longleftrightarrow 5$$

and generally,

$$k\pi \longleftrightarrow 2k \qquad \text{for } k > 0,$$

$$k\pi \longleftrightarrow -2k + 1 \qquad \text{for } k < 0.$$ ■

(continued from page 7)

and comparing it with the graphical representation of the solution set of the differential equation

$$Q'(t) - 1 = 0 \qquad (t \in R)$$

We know from Exercise 2 that the solution set of

$$\sin t = 0 \qquad (t \in R)$$

is the *set of numbers*

$$\{n\pi : n \in Z).$$

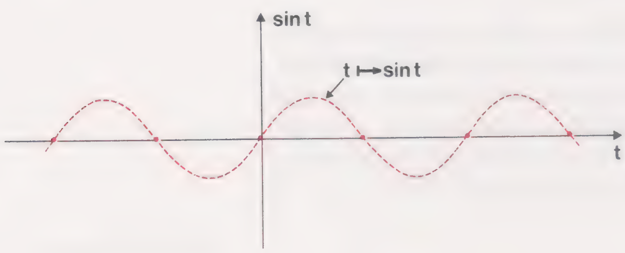
Graphically, this can be represented as the intersection of the straight line which is the pictorial graph of

$$t \longmapsto 0$$

and the curve which is the pictorial graph of

$$t \longmapsto \sin t.$$

The points of intersection have co-ordinates $(n\pi, 0)$, $n \in Z$.



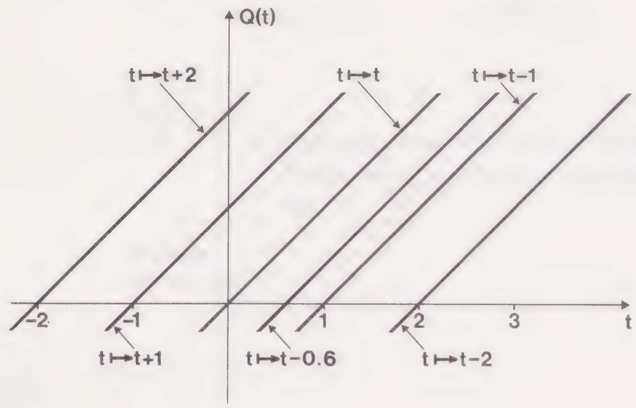
The solution set of the differential equation

$$Q'(t) - 1 = 0 \qquad (t \in R)$$

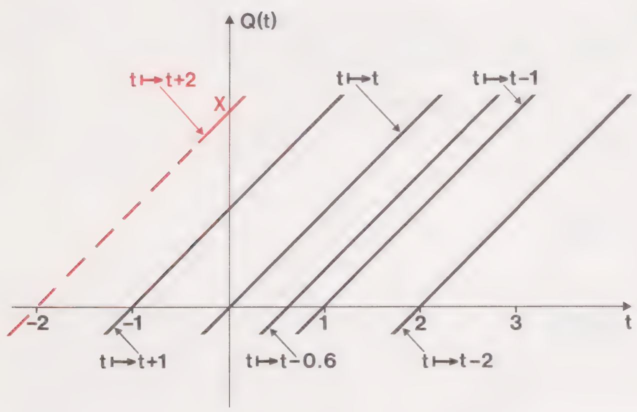
may be represented graphically by the *family of curves*, or set of solution curves, given by the pictorial graphs of the functions

$$Q : t \longmapsto t + c \qquad (t \in R), c \in R.$$

The fact that there is a *family* arises from all the different values that the constant of integration c may take. We illustrate the cases $c = 2, 1, 0, -0.6, -1, -2$.



The whole plane is filled by curves (straight lines in this case). Through any point, one, and only one, curve of the family passes. Picking out one point in the plane picks out one particular curve, which is equivalent to picking out a particular constant of integration. For example, consider a population of fish which starts ($t = 0$) with a population of 2 000 and increases at the constant rate of 1 000 per year (difference between number of births and number of deaths).



Then, using units of 1 000, we have an initial point (0, 2) (marked with an X on the diagram), and the appropriate solution curve (marked in red on the diagram) is the graph of

$$Q : t \longmapsto t + 2 \qquad (t \in R)$$

since our general solution (i.e. $Q : t \longmapsto t + c$) implies

$$Q(0) = 0 + c,$$

i.e.

$$2 = 0 + c.$$

In a differential equation obtained as a mathematical model of a real situation, we usually know an initial assigned value, as in this fishy example. Here, if the modelling is satisfactory, the one particular solution curve through the point

(initial time, initial population),

can be used to determine the future population. This process of picking a *particular curve* by the known *initial condition* is an important feature of the mathematical modelling in this situation.

Exercise 3

Exercise 3
(3 minutes)

Sketch a typical subset of the family of curves which illustrates the solution set of the differential equation

$$Q'(t) - 2t = 0 \qquad (t \in R).$$

Indicate the particular solution curve which satisfies the initial condition $Q(1) = 2$. What is the numerical value of the constant of integration then? ■

Exercise 4

Exercise 4
(2 minutes)

Rewrite

$$Q'(t) = 2t$$

in a form involving

(i) functions only,
(ii) variables only. ■

Discussion

A differential equation, like an ordinary equation, may have no members in its solution set. Thus

$$\{f : (f'(t))^2 + (f(t))^2 + 1 = 0 \qquad (t \in R)\}$$

is an empty set, just as

$$\{t : t^2 - 2t + 2 = 0 \qquad (t \in R)\}$$

is empty. We have just seen that the number of solutions of $Q'(t) - 2t = 0$ is not finite. Also a differential equation can have a finite number of functions in its solution set. Thus

$$\{f : (f'(t))^2 + (f(t))^2 = 0 \qquad (t \in R)\}$$

has one member,

$$f : t \longmapsto 0 \qquad (t \in R),$$

just as

$$\{t : t^2 - 2t + 1 = 0 \qquad (t \in R)\}$$

has only one member.

Summary

Summary

Solutions of differential equations are functions. The pictorial graphs of these solutions form a family of solution curves. In our example we used one initial condition (one point in the plane), to pick out one particular function (one particular solution curve).

24.2 SOME METHODS OF SOLVING DIFFERENTIAL EQUATIONS

24.2

24.2.0 Introduction

24.2.0

Introduction

In this section we look at three ways of finding solutions to first order differential equations. Each has its advantages and disadvantages and we shall point out some of these as we go along.

In section 24.2.1 we discuss a *graphical method* of solution and give an example.

In sections 24.2.2, 3 we pick out from the repertoire of *formula methods* of solution two which will give exact solutions to first order differential equations of certain types. (The solutions are exact in the sense that we can specify precisely the set of functions which form the solution set, rather than give them in tabulated form as with a numerical method.) The usefulness of formula methods lies in the fact that they produce a more "general" solution than the graphical method, just as the solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are written more generally as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

than by saying that if $a = 1$, $b = -2$ and $c = 1$, then $x = 1$, because the general solution can be evaluated for many different numerical values of a, b, c.

Why discuss only two formula methods? The reason is that the two we have chosen are simple enough to grasp and learn to use and are representative of the exact methods available. It is not our intention to list all the possible "recipes" for solving equations — after all, we do not intend that this should be a cookery book. Rather, we hope that you will gain a general idea of what is involved in finding exact solutions of differential equations, and will be able and prepared to learn other methods if and when you need them.

All the exact methods necessarily rely on the methods of integrating and differentiating functions we discussed in *Units 12* and *13*. We shall quote them explicitly as required.

Finally, in section 24.2.4 we give one example of a *numerical method* of solution.

24.2.1 Graphical Methods

24.2.1

Discussion

We wrote down the solution of the equation

$$\sin t = 0$$

straight away, since we knew that

$$\sin(n\pi) = 0 \qquad (n \in Z),$$

and no other values of t in the domain R of the sine function are mapped to zero. But suppose that we have a less tractable equation of the form $f(t) = 0$ to solve; for example, the equation we had for the "omelette problem" on page 34 of *Unit 2, Errors and Accuracy*:

$$t - \sin t - \frac{2\pi}{3} = 0.$$

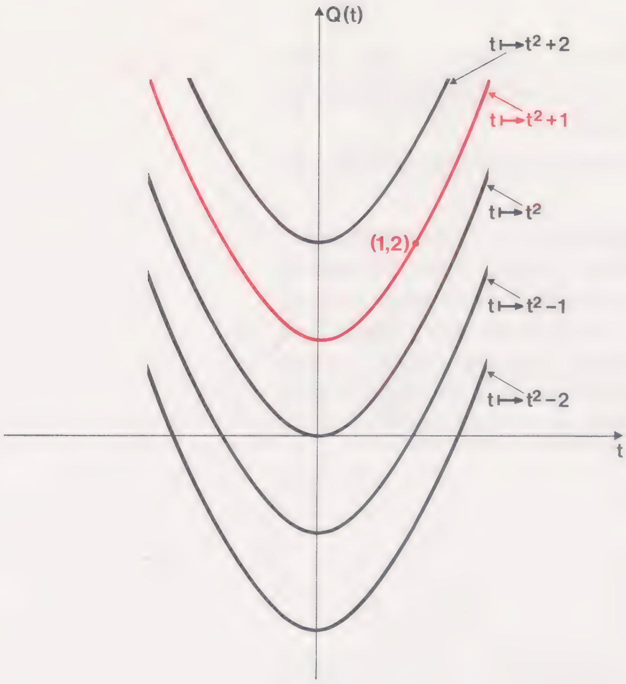
There we used graphical methods. We calculated the images of the function

$$t \longmapsto t - \sin t - \frac{2\pi}{3} \qquad (t \in R)$$

(*continued on page 13*)

Solution 24.1.2.3

Solution 24.1.2.3



The solution set is

$$\{Q : Q(t) = t^2 + c \qquad (t \in R), c \in R\}.$$

Using the initial condition

$$Q(1) = 2 = 1 + c$$

we obtain

$$c = 1$$

and the particular solution curve corresponds to

$$Q(t) = t^2 + 1.$$

The family of curves is, in this case, a family of parabolas. All the curves look similar in the same way that a biological family may have similar traits. That is one reason why it is convenient to use the word *family* to describe this particular collection of curves. ■

Solution 24.1.2.4

Solution 24.1.2.4

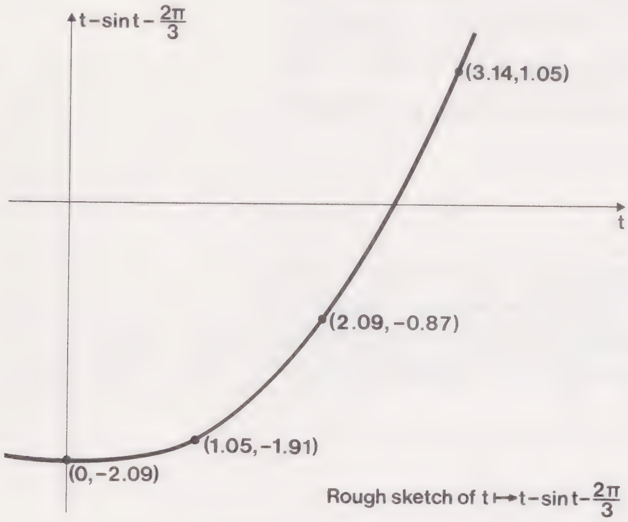
(i) $Q' = t \longmapsto 2t \qquad (t \in R)$

or $DQ = t \longmapsto 2t \qquad (t \in R)$

(ii) $\dfrac{dq}{dt} = 2t$ ■

(*continued from page 11*)

at a few selected points in the domain, joined them up smoothly (on the correct assumption that the function involved is continuous) and the value of t where the sketched curve crossed the x-axis was the approximate solution we wanted.



Rough sketch of $t \mapsto t - \sin t - \frac{2\pi}{3}$

To improve the accuracy, we effectively magnified the portion of the graph in the neighbourhood of the approximate solution, by calculating new images of the function and drawing that part of the graph more precisely.

We can adopt a similar approach for the solution of first order differential equations; this approach will be useful for those that we cannot solve by the standard methods that we discuss in this text. The last equation in the population growth example of section 24.1.1:

$$\frac{dq}{dt} = m_2 q(k_2 t + k_3 - q)$$

is one of this type. We shall however use the previous equation:

$$\frac{dq}{dt} = k_1 q - m_2 q^2$$

to illustrate the graphical method. Of course, to solve it graphically we must use actual numbers for k_1 and m_2. This illustrates the restrictive aspect of the graphical or numerical approach — not only do we have to specialize the problem in this way, but, in any case, we do not get a general expression for the solution. Suppose we choose $k_1 = 2$ and $m_2 = 1$. The differential equation is then

$$\frac{dq}{dt} = 2q - q^2$$

and we want to find the function, or functions, Q, such that $q = Q(t)$, which satisfy this equation. We shall be content to have such a function, or functions, specified graphically. We shall call the corresponding curves *solution curves.*

From *Unit 12, Differentiation I* we know that $\frac{dq}{dt} = Q'(t)$ represents the slope of the graph of the function Q, that is, the direction in which the solution curve is pointing at any point (t, q). For our differential equation we can calculate the slope of a solution curve at any point; it is always given by

$$2q - q^2.$$

Thus at the point $(0, 1)$, representing a population of 1 000 at time $t = 0$, the slope of the particular solution curve is

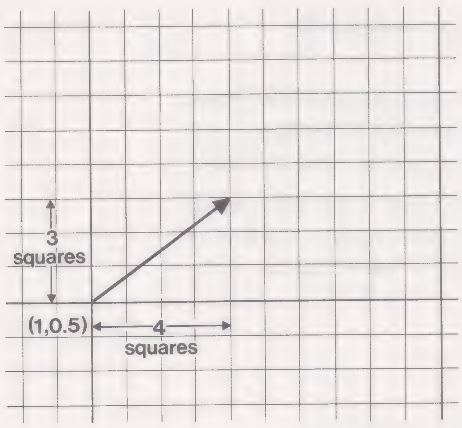
$$2 \times 1 - 1^2 = 1$$

Since $2q - q^2$ is not explicitly dependent on t, the slope at any point $(t, 1)$ is always 1. In terms of the population growth which the differential equation represents, this means, as we would expect, that the rate of increase is explicitly dependent on the size of the population and not the time at which we are measuring it. In terms of the solution curves, this means that for $q \geqslant 0$ we have a solution curve through every point (t, q), and for each q the tangents to the solution curves at (t, q) are parallel for all t.

Let us plot slopes at a selection of points on an appropriate graph. Remember that the slope is the ratio:

$$\frac{\text{vertical distance}}{\text{horizontal distance}}.$$

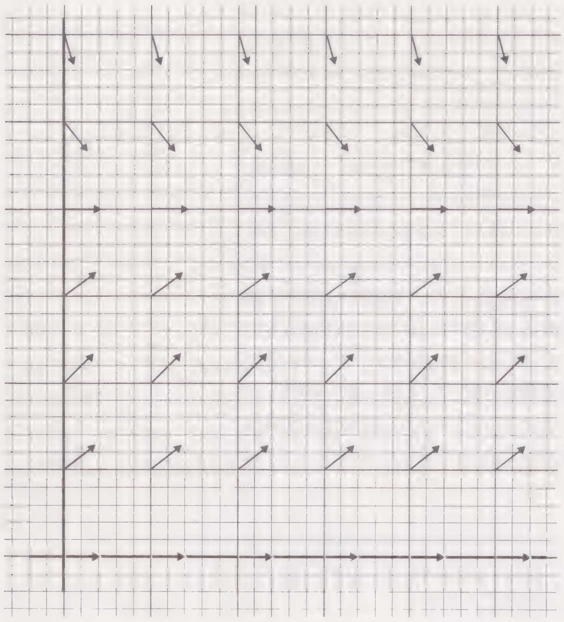
Let us therefore fix a horizontal distance of, say, 4 small squares on the graph paper, and then draw the arrow for a particular point to indicate the direction of the solution curve there. For example, for the point (1, 0.5) we need to draw an arrow to represent a slope of 0.75. The diagram indicates how this is done.



The actual numerical values of the slopes at some more selected points are displayed in the table below. As we have noted, the value of the slope does not depend on the value of t; so all the columns are the same. We have nevertheless set out a table, both to give you an idea of what happens in a more general case and to produce a record which is similar to the graphical picture.

3	−3	−3	−3	−3	−3	−3
2.5	−1.25	−1.25	−1.25	−1.25	−1.25	−1.25
2	0	0	0	0	0	0
1.5	0.75	0.75	0.75	0.75	0.75	0.75
1	1	1	1	1	1	1
0.5	0.75	0.75	0.75	0.75	0.75	0.75
0	0	0	0	0	0	0
q / t	0	0.5	1	1.5	2	2.5

If we draw the corresponding arrows, we obtain the following diagram:



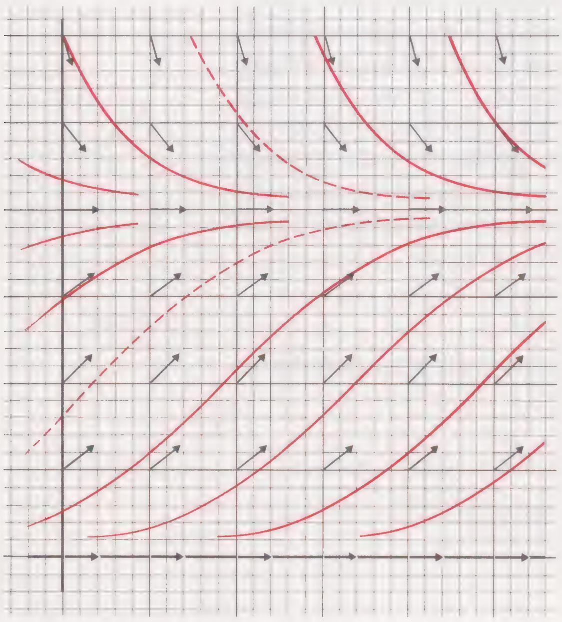
There are several points to note about the table and the diagram:

(i) Since all the numbers in any one row are the same, wherever solution curves cross a given horizontal line they must all be pointing in the same direction.

(ii) Normally we are interested only in that portion of the plane for which $q \geqslant 0$ and $t \geqslant 0$ (called *the first quadrant*) since this is the portion which is meaningful in the context of this problem.

(iii) We could go on to calculate the slope at more and more points in the first quadrant and make the set of arrows as numerous and dense as we please (physical width and length of pencil marks allowing).

(iv) In the same sense that we sketch a curve through a few points, so too we can sketch a curve, or curves, using the arrows as guides. The following diagram shows a few typical solution curves. All we have to do to sketch a solution curve is to start from a given point and make sure that the direction of the curve is roughly parallel to all the arrows it passes close to.



(v) If we wish to improve the accuracy, we must use more arrows, just as we would use more points to draw an ordinary graph.

(vi) What we have produced is an approximation to a selection of curves from the family of curves which represent the solution to the differential equation. We have thus, albeit laboriously, got the picture of what the family of solution curves look like. If you look at the set of arrows at a distance you get the impression of how the pattern "flows".

(vii) For our particular differential equation, we can see by substitution that $q = 0$ is a trivial solution: if the population is zero initially, mathematical and other reasoning tells us that the population will never be anything but zero. When $q > 0$, all the solution curves appear to approach the same numerical value 2 as t becomes large. 2 000 is thus the stable population figure. If there were more than 2 000 people originally they will reduce to this number: if less they will increase to this number. As we mentioned in the last section, we could have deduced this *particular* piece of information *without solving* the differential equation. For the derivative, $Q'(t)$, is zero when

$$2q - q^2 = 0,$$

that is when $q = 0$ (and we have dispensed with that) and when $q = 2$. The value $q = 2$ is never attained in our present model except in the special case when the population starts with 2 000.

(viii) To select the *particular* solution curve representing a particular case of interest, we need more information; e.g. an initial condition. Suppose we start with 750 in the population; that is, $q = 0.75$, and choose to measure time from that instant, so that $q = 0.75$ when $t = 0$. We start to draw our curve from this point and produce the approximate solution curve indicated by a red dotted line in the previous diagram. After 2 years* we see (from our curve) that the population of 2 000 has almost been attained. After one year we estimate that there would be 1 700 in the population.

The graphical method works for all first order differential equations which we can put in the form

$$\frac{dq}{dt} = f(t, q),$$

with certain conditions on f. All we have to do is to evaluate the images under f at appropriate points in the plane. The conditions on f which ensure that a solution exists are too complicated for the Foundation Course and will be dealt with in a later course.

The graphical method can be a very useful method in cases which cannot be solved by any other means. It can be inaccurate (as can curve-sketching of any type) and it is certainly laborious, but one advantage is that we get a good qualitative idea of the shapes of all the solution curves, that is, the solution set as a whole. With this "panoramic view" we can choose particular points of interest and find numerically the particular solution curves which pass through them. Another advantage of the graphical method is that it gives an introduction to numerical methods of solution of differential equations. We look at these in section 24.2.4.

Summary

Summary
* *

In the graphical approach we use the geometric information given by the differential equation

$$\frac{dq}{dt} = f(t, q),$$

namely the slopes of the solution curves at all the points (t, q) of the domain of f, to sketch the family of solution curves.

* Better late than never!
Were you quicker than the author?

24.2.2 Formula Method 1: Separation of Variables

24.2.2

Main Text

This method is useful only when we can "disentangle" the variables in the equation.

We illustrate the method by an example. Consider the equation

$$Q'(t) = tQ(t) \qquad (t \in R).$$

We regard t and $Q(t) = q$ as variables, and we can "disentangle" them by rearranging the equation in the form

$$\frac{Q'(t)}{Q(t)} = t$$

provided that we exclude any t such that $Q(t) = 0$ from the domain of Q. Writing this in function (rather than image) form we get

$$\frac{1}{Q} \times DQ = t \longmapsto t \qquad (t \in R).$$

Since $\frac{1}{Q} \times DQ$ and $t \longmapsto t$ are equal functions, we can equate their primitives provided that we choose the constants of integration correctly. We therefore try to "integrate both sides".

We can integrate the right-hand side to get the set of primitive functions of the form $t \longmapsto \frac{t^2}{2} + c, c \in R$, but the integral of the left-hand side may not be so obvious.* In *Unit 13, Integration II* (section 13.2.4) we obtained the following formula for the primitive function of a composite function:

$$\int (g \circ k) \times Dk = \left(\int g\right) \circ k$$

Comparing this with our equation, we see that we need to choose $k = Q$ and $g \circ k = \frac{1}{Q}$, that is

$$g : t \longmapsto \frac{1}{t}$$

Then from our table of integrals we have

$$\int g = t \longmapsto \ln t + c_1 \qquad (t \in R^+)$$

and

$$\left(\int g\right) \circ k = t \longmapsto \ln Q(t) + c_1 \qquad (Q(t) \in R^+),$$

where c_1 is any real number. Notice how another condition has crept in here, that is, $Q(t) > 0$ (because the logarithm function has domain R^+). We could deal with $Q(t) < 0$ if it were of any interest, but we would have to do it separately. When using standard results, as we are here, we must always be careful to use them in the appropriate circumstances.

Putting the bits together, we have

$$t \longmapsto \ln Q(t) + c_1 = t \longmapsto \frac{t^2}{2} + c$$

This specifies the set of functions Q which belong to the solution set of the original equation, but in an inconvenient form: the dependent variable $q = Q(t)$ is not expressed explicitly in terms of the independent variable t.

* In fact, we are on familiar ground; see Exercise 2, section 12.3.2 of *Unit 12*.

So we try to reorganize. First we use the simpler form in terms of images:

$$\ln q + c_1 = \frac{t^2}{2} + c$$

i.e.

$$\ln q = \frac{t^2}{2} + (c - c_1)$$

Since c and c_1 are any real numbers, $c - c_1$ is any real number. We can, therefore, either write $d \in R$ for $c - c_1$ or just drop the c_1. (In general, we do not need to introduce the constant of integration on both sides of an equation, since we know that if two functions are equal, their primitives differ by a constant.) So now we have

$$\ln q = \frac{t^2}{2} + d$$

Remembering that ln is a one-one function and exp is its inverse, we have

$$q = \exp\left(\frac{t^2}{2} + d\right)$$

We now have the dependent variable $q = Q(t)$ expressed explicitly in terms of the independent variable t, and we could say that we have a satisfactory solution to our problem. In fact, we can tidy things up a little further if we remember that

$$\exp x = e^x$$

So

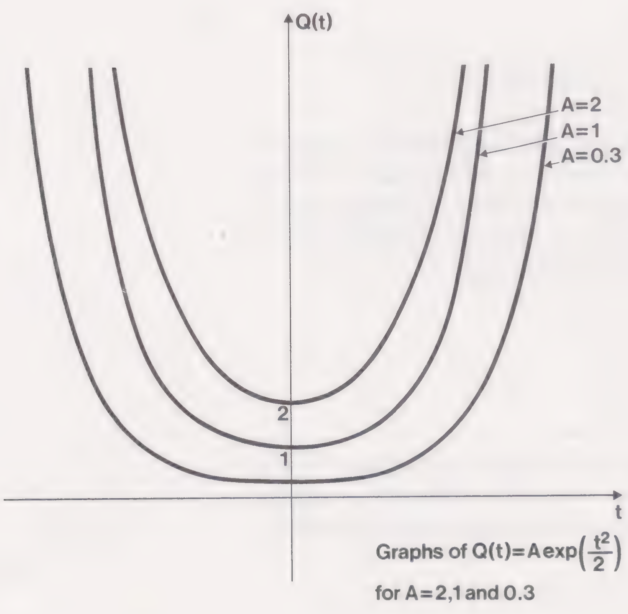
$$\exp\left(\frac{t^2}{2} + d\right) = e^{(t^2/2)+d} = e^{t^2/2} \times e^d$$

If we write $A = e^d$, the general solution becomes

$$q = Ae^{t^2/2} \text{ or } A \exp\left(\frac{t^2}{2}\right),$$

where $A \in R^+$, since $\exp d$ cannot be negative or zero.

The family of solution curves is illustrated below.



Graphs of $Q(t)=A\exp\left(\frac{t^2}{2}\right)$ for A=2,1 and 0.3

We have solved this equation in small steps, and it is worth noticing that we have sprinkled conditions on the way. Originally we required $Q(t) \neq 0$, and then we required $Q(t) > 0$, which superseded the first condition. Finally, we have

$$Q(t) = A \exp\left(\frac{t^2}{2}\right)$$

and since $A \in R^+$, the condition $Q(t) > 0$ is automatically satisfied. Also notice how the constants of integration, c and c_1, which originally appeared added on to the ends of our expressions and were unrestricted, became "entangled" as a result of our manipulations, so that our final constant, A, *is* qualified ($A > 0$). With practice, the formal manipulation can become deceptively easy, but the most adept manipulators often forget to consider the consequences of what they are doing.

And yet, having been so ponderously careful, what if we had not been so careful? You can easily check that

$$Q(t) = A \exp\left(\frac{t^2}{2}\right)$$

with $A \in R^-$, also belongs to the solution set of the original differential equation. (The solution curves are similar to those described in the last figure but they are all upsidedown below the t-axis.) So perhaps a little carelessness pays. Well, it may do, just as in any other scientific field; but it must be *disciplined* carelessness. In this case, the discipline lies in checking that our formal manipulations, without conditions, have not caused us to include in our solution set functions which are not in fact solutions.

To generalize the approach using the separation of variables, we need to be able to recast the original differential equation in the form

$$(g \circ Q) \times DQ = h,$$

Equation (1)

or, in terms of variables (images):

$$g(q)\frac{dq}{dt} = h(t),$$

and then we hope to be able to integrate. Herein lies the reason for the name of the method; we "separate" the q's to one side of the equation to enable us to integrate. This means that the original differential equation

$$\frac{dq}{dt} = f(t, q)$$

must be expressible in the form

$$\frac{dq}{dt} = \frac{h(t)}{g(q)}$$

Exercise 1

Exercise 1
(2 minutes)

There are some fairly sophisticated conditions on g and h for this method to work, which we shall not discuss in this text; but what *obvious* restriction must we place on g? ■

Main Text

Returning to the form in Equation (1), we have

$$(g \circ Q) \times DQ = h$$

Using the result for integrating a composite function quoted above, we get

$$\int (g \circ Q) \times DQ = \int h + (t \longmapsto c)$$

(*continued on page 20*)

Solution 1

Solution 1

$g(q)$ may not be zero, i.e. we must exclude those values of t for which $g(Q(t)) = 0$. ■

(*continued from page 19*)

that is

$$\left(\int g\right) \circ Q = \int h + (t \longmapsto c)$$

Thus, if we can obtain $\int g$ and $\int h$ from appropriate tables (perhaps after further manipulation), we can solve the differential equation.

Summary

Summary
★ ★ ★

If

$$DQ = \frac{h}{g \circ Q} \qquad (g(Q(t)) \neq 0, t \in R),$$

then the solutions satisfy the equation

$$\left(\int g\right) \circ Q = \int h + (t \longmapsto c) \qquad c \in R.$$

The Leibniz form of the separation of variables rule is outlined below.

If

$$\frac{dq}{dt} = \frac{h(t)}{g(q)} \qquad (g(q) \neq 0),$$

then the solutions satisfy the equation

$$\int g(q)\, dq = \int h(t)\, dt + c$$

This is a form you will probably find in textbooks; you may find this form easier to remember.

Exercise 2

Exercise 2
(3 minutes)

Find the function Q which is a solution of

$$q\frac{dq}{dt} = 2t \qquad (t \in R)$$

where $q = Q(t)$, and which satisfies the initial condition

$$Q(0) = 3$$ ■

Exercise 3

Exercise 3
(3 minutes)

Find the solution set of

$$\frac{dq}{dt} = \frac{-kq}{q + a},$$

where $q = Q(t)$, and a and k are positive numbers. State any conditions you find necessary in the manipulations. ■

24.2.3

Main Text

24.2.3 Formula Method 2: Integrating Factor

We can now find formula solutions for first order differential equations of two types

(i) $Q'(t) = g(t)$,

by straightforward integration, and

(ii) $Q'(t) = \dfrac{g(t)}{h(Q(t))} \qquad h(Q(t)) \neq 0,$

by separating the variables, *provided* the resulting integrals can be found with reasonable ease. Obviously there will be cases where the equation does not admit of the particular form required in the separation of variables method. For example,

$$Q'(t) = -Q(t) + t^2 \exp(-t)$$

cannot be put in the required form. We can separate the variables in the sense that we can get all the dependent variables on one side, i.e.

$$Q'(t) + Q(t) = t^2 \exp(-t),$$

but we cannot get the left-hand side into the form $(g \circ Q) \times DQ$. However, we can adopt the same idea. Our separation of variables method relied on our being able to use known results about the integration of composite functions, and these were themselves obtained from our result for differentiating composite functions. Now we also know how to differentiate the sum and product of functions. The "sum" result is so obvious that it is unlikely to give us any technique for solving differential equations which is not itself obvious. But the "product" result (*Unit 12, Differentiation I*, section 12.2.4)

$$D(f \times g) = (Df) \times g + f \times (Dg)$$

is not so obvious. Suppose that we assume that the left-hand side of our differential equation is of the form $D(f \times g)$, for some choice of f and g. Then

$$(Df) \times g + f \times (Dg) = Q' + Q$$

Now we can choose $f = Q$; then we have

$$Q' \times g + Q \times (Dg) = Q' + Q$$

i.e. in image form

$$Q'(t)(g(t) - 1) + Q(t)(Dg(t) - 1) = 0$$

If Q and Q' are linearly independent,* then

$$\alpha Q + \beta Q' = t \longmapsto 0 \Leftrightarrow \alpha = \beta = 0$$

i.e.

$$\alpha Q(t) + \beta Q'(t) = 0 \Leftrightarrow \alpha = \beta = 0$$

In this case, we have no further choice, in the sense that

$$Dg = t \longmapsto 1 \qquad (t \in R)$$

and

$$g = t \longmapsto 1 \qquad (t \in R)$$

and this is plainly impossible. However, now that we have started on this way of thinking, let's continue with it. We obtained the form

$$Q'(t) + Q(t) = t^2 \exp(-t)$$

by separating out the terms involving the dependent variable $q = Q(t)$;

* See *Unit 22*, section 22.2.2.

(*continued on page 23*)

Solution 24.2.2.2

Solution 24.2.2.2

The equation is already in separated form. Comparing it with the equation

$$\frac{dq}{dt} = \frac{h(t)}{g(q)}$$

we see that

$$g(q) = q, h(t) = 2t$$

Therefore suitable primitive functions of g and h respectively are

$$\int q \longmapsto q = q \longmapsto \frac{q^2}{2} \quad \text{and} \quad \int t \longmapsto 2t = t \longmapsto t^2$$

So the solution set is

$$\left\{Q : \frac{q^2}{2} = t^2 + c \qquad (t \in R), c \in R\right\}.$$

The particular value of c determined by the initial condition is given by

$$\tfrac{1}{2}(Q(0))^2 = 0^2 + c, \text{ where } Q(0) = 3,$$

i.e.

$$c = \tfrac{9}{2}$$

Therefore the particular solution we are interested in is given by

$$\frac{(Q(t))^2}{2} = t^2 + \frac{9}{2}$$

But this determines two possible functions Q:

(i) $Q : t \longmapsto \sqrt{2t^2 + 9}$

(ii) $Q : t \longmapsto -\sqrt{2t^2 + 9}$

Remembering again that $Q(0) = 3$, i.e. $Q(0) > 0$, we see that the required solution is

$$Q : t \longmapsto \sqrt{2t^2 + 9}$$ ■

Solution 24.2.2.3

Solution 24.2.2.3

A first condition is obviously that $q + a \neq 0$, i.e. $Q(t) \neq -a$. In separated form the equation becomes

$$\frac{q + a}{q}\frac{dq}{dt} = -k,$$

and we have to impose the further condition $q \neq 0$, i.e. $Q(t) \neq 0$. Referring back to the summary, we have

$$g(q) = \frac{q + a}{q} = 1 + \frac{a}{q}, h(t) = -k$$

Suitable primitive functions are

$$\int g = q \longmapsto q + a \ln q \qquad (\text{assuming } q \in R^+),$$

$$\int h = t \longmapsto -kt$$

Therefore, the solution set is

$$\{Q : Q(t) + a \ln Q(t) = -kt + c \qquad (t \in R), c \in R, Q(t) \in R^+\}$$

Notice that $Q(t) \in R^+$ takes care of the previous restrictions on $Q(t)$. This time we cannot get the solution set in an explicit form, that is, we

cannot get q, the dependent variable, explicitly in terms of t, the independent variable, and thus it is difficult to determine $Q(t)$ for a given t. In fact, we could use one of the iterative methods we introduced in *Unit 2, Errors and Accuracy*, or evaluate $Q(t)$ at a selection of values of t and then use interpolation to find other values of $Q(t)$. On such occasions it may be just as simple to solve the differential equation by a numerical method straight away. ■

(*continued from page 21*)

and then we tried to see if the left-hand side had product form. The latter part is the important bit, so let's concentrate on that. Can we find a left-hand side, variables separated or not, such that we can recognize a product form? Assuming that one function in the product is Q, say f as above, then we require a form

$$Q' \times g + Q \times Dg.$$

Comparing this with what we have actually got in the differential equation, we must somehow find a way of multiplying Q' by a function g, and Q by its derivative Dg. Now we can't multiply Q' by one function and Q by another without changing the differential equation: we must multiply them both (and of course the right-hand side) by the *same* function. This function must therefore satisfy the equation

$$Dg = g$$

Now any function g satisfying this equation will do, and we know one solution, the exponential function, since

$$D(t \longmapsto \exp t) = t \longmapsto \exp t$$

So we multiply both sides of our equation by $\exp t$ (which luckily is not zero anywhere, so we are not likely to produce nonsense). We obtain

$$(\exp t)Q'(t) + (\exp t)Q(t) = t^2(\exp t)(\exp(-t)) = t^2$$

i.e.

$$D(\exp \times Q) = t \longmapsto t^2,$$

or

$$\exp \times Q = t \longmapsto \frac{t^3}{3} + c,$$

from which we can get Q or $Q(t) = q$ explicitly as

$$q = \exp(-t) \times \left(\frac{t^3}{3} + c\right)$$

So the solution set is

$$\left\{Q : Q(t) = \exp(-t) \times \left(\frac{t^3}{3} + c\right) \quad (t \in R), c \in R\right\}.$$

Before we discuss some general aspects of this method we suggest you try it for yourself in the following exercise.

Exercise 1

Exercise 1
(4 minutes)

Find the solution set of the equation

$$\frac{dq}{dt} + \frac{q}{t+a} = t \qquad (t \in R, t \neq -a)$$

where a is a positive number and $q = Q(t)$. ■

Solution 1

Solution 1

As before, we assume that we can get the left-hand side of the equation in the form of a derivative of a product, i.e. we assume that it can take the form

$$\frac{d}{dt}(q \times g(t)) = \frac{dq}{dt}g(t) + q\frac{d(g(t))}{dt}$$

To get the left-hand side of our differential equation into this form, we must multiply $\frac{dq}{dt}$ by $g(t)$ and $\frac{q}{t+a}$ by $\frac{d(g(t))}{dt} \times (t+a)$. These two must be the same, i.e. we need a function g such that

$$(t+a)\frac{d(g(t))}{dt} = g(t)$$

We can separate the variables in this equation in the sense of our first mėthod, to obtain

$$\frac{1}{g(t)}\frac{d(g(t))}{dt} = \frac{1}{t+a}$$

We are looking for any one solution of this equation so we take the constant of integration to be zero.
(We have seen the left-hand side before.)
Integrating, we get one solution

$$\ln(g(t)) = \ln(t+a) \qquad (t+a>0)$$

or

$$g(t) = t + a$$

So multiplying both sides of the original differential equation by $t + a$ (which again is not zero, since $t + a > 0$), we get

$$(t+a)\frac{dq}{dt} + q = t(t+a)$$

(Of course, this rearrangement may have been obvious to you from the start: we have given a formal approach just in case it wasn't.) We have

$$\frac{d}{dt}(q(t+a)) = t^2 + at,$$

so that

$$q(t+a) = \frac{t^3}{3} + \frac{at^2}{2} + c,$$

which gives the solution set

$$\left\{Q : Q(t) = \frac{1}{t+a}\left(\frac{t^3}{3} + \frac{at^2}{2} + c\right)(t \in R, t \neq -a), c \in R\right\}.$$

(In the process of solution we required $t + a > 0$, but this condition is not essential: it is easy to verify that the above describes the solution set for $t + a < 0$ as well.) ■

Discussion
* *

The crucial step in both the example in the text and Exercise 1 is the multiplication by a suitable function g, so that the left-hand side of the equation can be written as the derivative of the product of g with Q, i.e.

$$D(g \times Q) = gQ' + g'Q$$

If we are going to adopt this as the general form, then it implies that we are dealing with an equation of the form

$$Q' + \frac{g'}{g}Q = \frac{h}{g}, \qquad g(t) \neq 0,$$

except that we are not going to be told what g is.

So let's start with a given equation of the general form

$$Q' + PQ = R,$$

where P and R are known functions. Comparing this with the previous equation, we see that g is determined by

$$\frac{g'}{g} = P$$

Remembering that we are looking for just one solution, we ignore the constant of integration, and get

$$\ln \circ g = \int P \qquad g(t) \in R^+$$

or

$$g = \exp \circ \int P$$

or

$$g(t) = \exp\left(\int P(t)\,dt\right)$$

Summarizing, we get the following rule:

Rule
* * *

To solve an equation of the form

$$Q'(t) + P(t)Q(t) = R(t),$$

where P and R are known functions, multiply throughout by the *integrating factor*

$$g(t) = \exp\left(\int P(t)\,dt\right)$$

and integrate directly.

Let us try an example. We shall find the solution set of the equation

$$Q'(t) + 2tQ(t) = 4t^3 \exp(-t^2) \qquad (t \in R)$$

To compare with the rule, we take

$$P: t \longmapsto 2t$$

and a simple integral is

$$t \longmapsto t^2.$$

Therefore an integrating factor is

$$g(t) = \exp\left(\int P(t)\,dt\right) = \exp(t^2) = e^{t^2},$$

and we have

$$\begin{aligned}\frac{d}{dt}(e^{t^2}Q(t)) &= e^{t^2}Q'(t) + 2te^{t^2}Q(t) \qquad \text{(derivative of a product)}\\ &= e^{t^2}(Q'(t) + 2tQ(t))\\ &= e^{t^2}(4t^3e^{-t^2}) \qquad \text{(from given differential equation)}\\ &= 4t^3\end{aligned}$$

(Note that the left-hand side is $\frac{d}{dt}$(integrating factor $\times$ $Q(t)$).)

The integral is then

$$Q(t)e^{t^2} = t^4 + c,$$

and the solution set may be written as

$$\{Q : Q(t) = e^{-t^2}(t^4 + c) \qquad (t \in R)\, c \in R\}$$

Exercise 2

Exercise 2
(4 minutes)

Indicate which of the following differential equations can be solved (assuming that the appropriate primitive functions can be found for the integrals involved), by

A: both the method of separation of variables and the integrating factor method;
B: the method of separation of variables and not the integrating factor method;
C: the integrating factor method and not the method of separation of variables;
D: neither the integrating factor method nor the method of separation of variables.

(i) $\dfrac{dq}{dt} = \ln(tq) \qquad (t \in R^+, q = Q(t) \in R^+)$

(ii) $(\exp t)\dfrac{dq}{dt} = q^2$

(iii) $Q'(t) = \sqrt{t}Q(t) \qquad (t \in R^+, Q(t) \in R^+)$

(iv) $\dfrac{dq}{dt} = \cos q + \cos t$

(v) $\dfrac{dq}{dt} = \dfrac{q^2 + 1}{t^2 + 1}$

Solve two of the above equations for which you can find the primitive functions for the integrals involved. ■

The remainder of this section can be omitted if you are short of time.

Example
*

As a last example of a formula method, we solve the population growth equation (see page 5):

$$\frac{dq}{dt} = k_1 q - m_2 q^2 \qquad (t \in R, q \in R^+)$$

where $q = Q(t)$, and k_1 and m_2 are positive numbers.

Neither the integrating factor method, nor the method of separation of variables is directly appropriate, but there are many other methods available.

We shall replace the function Q by the function U, where

$$Q(t) = \frac{1}{U(t)}$$

i.e.

$$q = \frac{1}{u},$$

where $u = U(t)$.

If you are familiar with the Leibniz notation, you can deduce from the last equation that

$$\frac{dq}{dt} = \frac{-1}{u^2}\frac{du}{dt}$$

(Alternatively, using the notation of *Unit 12, Differentation I*, you could deduce the same result from $Q = \left(t \longmapsto \frac{1}{t}\right) \circ U$.)

Substituting for $\frac{dq}{dt}$ and q in the differential equation, we get

$$-\frac{1}{u^2}\frac{du}{dt} = \frac{k_1}{u} - \frac{m_2}{u^2};$$

that is,

$$\frac{du}{dt} = -k_1 u + m_2$$

or

$$\frac{du}{dt} + k_1 u = m_2$$

This equation can now be solved by the integrating factor method. An integrating factor is

$$g(t) = \exp\left(\int k_1\, dt\right) = \exp(k_1 t),$$

so the differential equation may be written:

$$\frac{d}{dt}(\exp(k_1 t)u) = m_2 \exp(k_1 t)$$

and after integrating we obtain

$$\exp(k_1 t)u = \frac{m_2}{k_1}\exp(k_1 t) + c$$

or

$$u = \frac{m_2}{k_1} + c\exp(-k_1 t) \qquad (t \in R)$$

Since $q \in R^+$, $u \in R^+$, so that the constant c cannot be negative. Having solved the equation, we replace u by $\frac{1}{q}$ to obtain the result in terms of our original variable q.

$$\frac{1}{q} = \frac{m_2}{k_1} + c\exp(-k_1 t)$$

The explicit form of the solution set is thus

$$\left\{Q : Q(t) = \frac{1}{\dfrac{m_2}{k_1} + ce^{-k_1 t}} \qquad (t \in R_0^+), c \geqslant 0\right\}.$$

In this case we can determine the population at a specified time t by reference to exponential tables rather than by drawing a solution curve.

Summary

Summary
* *

In this and the preceding section we have described two formula methods of solving first order differential equations: the "separation of variables" and "integrating factor" methods. If you are faced with the need to solve a first order differential equation which is not amenable to one of these methods, you can either refer to other formula methods described in standard texts (see Bibliography), or use the graphical method or a numerical method. The next section is devoted to introducing numerical methods.

Solution 2

Solution 2

(i) D

(ii) B

Solution set is

$$\left\{Q : Q(t) = \frac{1}{\exp(-t) + c} \qquad (t \in R), c \in R\right\}$$

If c is negative, there will be the problem that $Q(t)$ is not defined for one value of t and so there would be a discontinuity in Q.

(iii) A
Solution set is

$$\{Q : Q(t) = \exp\left(\tfrac{2}{3}t^{3/2} + c\right) \qquad (t \in R^+), c \in R\}$$

or

$$\{Q : Q(t) = a \exp\left(\tfrac{2}{3}t^{3/2}\right) \qquad (t \in R^+), a \in R^+\}$$

(iv) D

(v) B
To find the solution by the method of separation of variables requires us to be able to recognize the primitive function

$$\int t \longmapsto \frac{1}{t^2 + 1}$$

We have not formally covered this in the course: it is, in fact,

$$t \longmapsto \arctan t \qquad (t \in R)$$

which is defined to be the inverse function of

$$t \longmapsto \tan t \qquad \left(t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right).$$

(See Exercise 1, section 13.2.6 of *Unit 13*.) The solution set is then

$$\{Q : \arctan Q(t) = \arctan t + c \qquad (t \in R), c \in R\}.$$

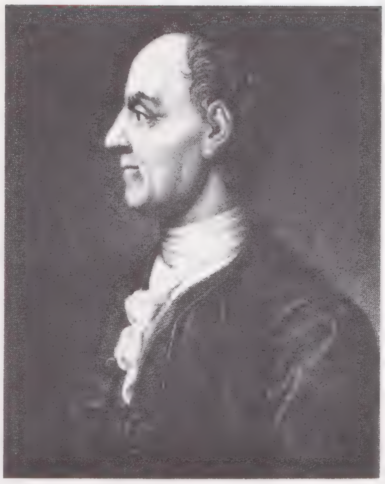
This could be simplified if there were any further interest in the problem. ■

24.2.4 A Numerical Method of Solution

24.2.4

Main Text

There are many powerful numerical methods of solving differential equations. As with the other methods described earlier in this text, we want to give you some idea of just one such numerical method. It is not very accurate, but it will give you an idea of the basis of the numerical approach. It is a method which, if suitably refined, could well be used to solve a differential equation. It is called Euler's Method. The basic idea of the method is rather like hunting for pirate's treasure.



Leonhard Euler
1707–1783

> Go to the solitary tree on the top of the hill.
> Take 20 paces in an easterly direction.
> Then take 15 paces in a north-easterly direction.
> Etc.

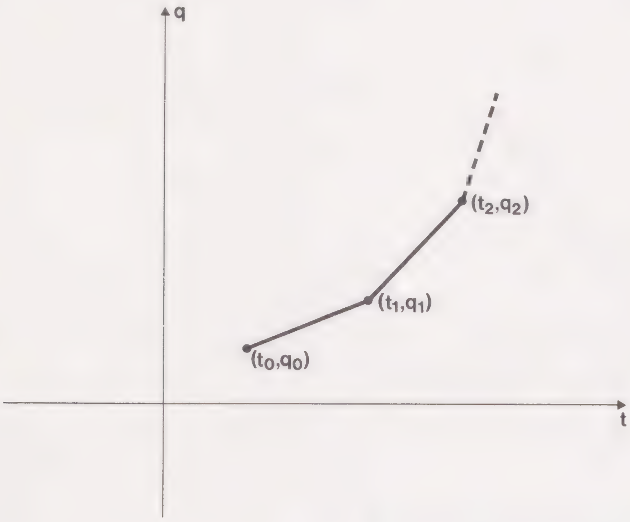
Euler's method can be used to solve an equation of the form

$$Q'(t) = f(t, q),$$

with given initial condition $Q(t_0) = q_0$, where $q = Q(t)$, f is a known function of two (real) variables, and t_0 and q_0 are known numbers.

(t_0, q_0) is the pair of co-ordinates of the point (the solitary tree on the top of the hill) from which the solution curve starts.

The differential equation tells us the direction in which to set off; we evaluate $f(t_0, q_0)$, and that is $Q'(t_0)$.



In Euler's method we can decide how far we should go in that direction before recalculating a new direction. Intuitively, the shorter the distance we go, the more accurate our estimate of the solution curve is likely to be. The first step takes us to some new point (t_1, q_1) where $Q(t_1) = q_1$.

Again the differential equation will tell us in which direction to go, since

$$Q'(t_1) = f(t_1, q_1)$$

is the slope of the solution curve through the point (t_1, q_1). We can continue for as long as we wish. The resulting path consists of straight line segments, which we hope are a good approximation to a part of the true solution curve through the initial point.

Let's see how this method works by applying it to a particular example. In doing this we have a choice to make. We could choose an example of a differential equation which cannot be solved any other way, or we could choose an example of a differential equation for which the solution is already known. We have chosen the latter because the former tends to

involve more complicated arithmetic. Also, using the latter for first acquaintance with the method, we can see how well we are doing and get the feel of what is going on. We shall estimate the solution curve to

$$Q'(t) = 2Q(t)$$

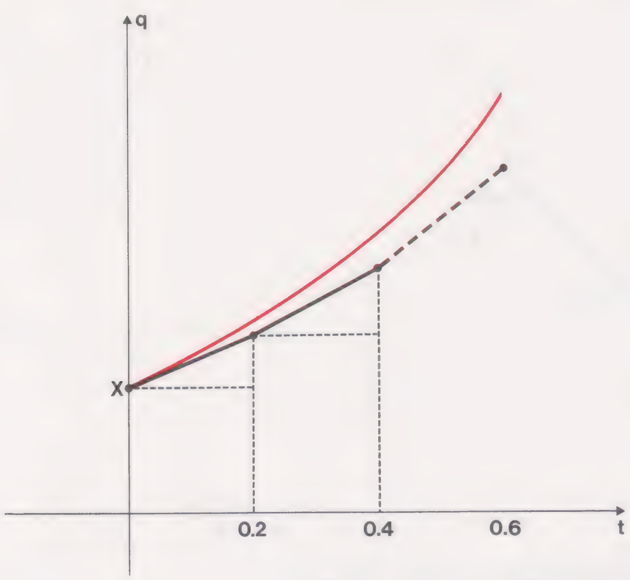
with the initial condition

$$Q(0) = 0.2,$$

i.e. the required solution curve passes through the point (0, 0.2). We can in fact solve this exactly, either by using the "separation of variables" method or the "integrating factor" method (or almost from foreknowledge!): the required solution curve is the pictorial graph of the function

$$Q_1 : t \longmapsto 0.2 \exp(2t)$$

We shall now work on the following figure, on which the correct solution curve has already been drawn in red, and use a numerical method of solution. The "solitary tree" we start from is marked by a cross on the graph paper at the point (0, 0.2). We now have to choose the number of "paces" or "distance in a fixed direction" we are going to take. Unlike the treasure hunt, we do not measure the paces along the path but take a fixed step length in the t-direction. (This is the same as in the other approximation methods we used in *Unit 4, Finite Differences*, when using finite differences to interpolate, and in *Unit 9, Integration I* when evaluating definite integrals using the trapezoidal or Simpson's rule.)



We shall use a step length of 0.2. The direction in which we have to go at the point (0, 0.2) is obtained from the differential equation. It is along a straight line path whose slope is

$$Q'(0) = 2 \times (0.2) = 0.4$$

We continue along this path until we have covered our step length on the t-axis, i.e. until the t value is 0.2. (We draw it by measuring 4 squares up from the horizontal line through (0, 0.2) for every 10 squares along it.)

Suppose that this path we are producing is the pictorial graph of a function V, then

$$\begin{aligned} V(0) &= 0.2 \\ V(0.2) &= 0.2 + (\textit{slope of straight line segment}) \times (0.2) \\ &= 0.2 + 0.4 \times 0.2 = 0.28 \end{aligned}$$

To calculate the new direction in which to go from the point (0.2, 0.28), we calculate the slope of the solution curve through (0.2, 0.28). Note that we are already "off" the actual solution curve: our change in direction at this point is to try to keep "close" to this curve. At (0.2, 0.28),

$$\begin{aligned} Q'(0.2) &= 2Q(0.2) \\ &= 2V(0.2) \\ &= 2 \times 0.28 = 0.56 \end{aligned}$$

We therefore continue the graph of V by following a straight line segment from (0.2, 0.28) with a slope of 0.56 until we reach a t value of 0.4. Then

$$\begin{aligned} V(0.4) &= 0.28 + 0.56 \times 0.2 \\ &= 0.392 \end{aligned}$$

and we have determined the next "corner" of the graph of V.

Continuing the procedure we would find

$$\begin{aligned} V(0.6) &= 0.392 + Q'(0.4) \times 0.2 \\ &= 0.392 + 2V(0.4) \times 0.2 \\ &= 0.392 + 2 \times 0.392 \times 0.2 \\ &\simeq 0.549 \end{aligned}$$

Exercise 1

Exercise 1
(5 minutes)

By using a step length of 0.1 instead of 0.2, determine the first four line segments of the estimated corresponding solution curve, call it V_1, and sketch it on a diagram. Does the graph of V_1 appear to be "better" than the graph of V, in the sense that it is nearer to the graph of the actual solution? ■

Exercise 2

Exercise 2
(5 minutes)

You may be wondering how to determine the estimated solution curve for $t < t_0$, that is, $t \in R^-$ in this case. How would you determine $V(-0.2)$? ■

Now let us generalize and suppose that we wish to find an approximate solution to

$$Q'(t) = f(t, q)$$

for t in the interval $[a, b]$, given the initial condition

$$Q(a) = q_0$$

For simplicity, we divide the total interval into n equal step lengths

$$h = \frac{b - a}{n}$$

and label the t values

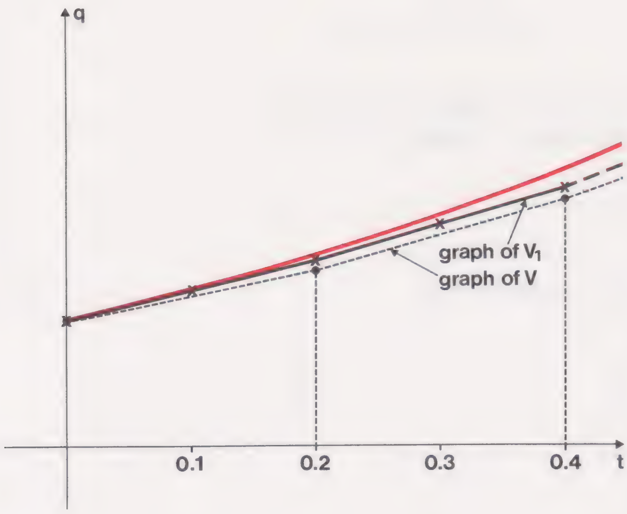
$$\begin{aligned} t_0 &= a \\ t_1 &= a + h \\ &\vdots \\ t_r &= a + rh \\ &\vdots \\ t_n &= a + nh = b \end{aligned}$$

(continued on page 33)

Solution 1

Solution 1

The tabular points for V_1 are approximately (0, 0.2), (0.1, 0.24), (0.2, 0.288), (0.3, 0.346), (0.4, 0.415).

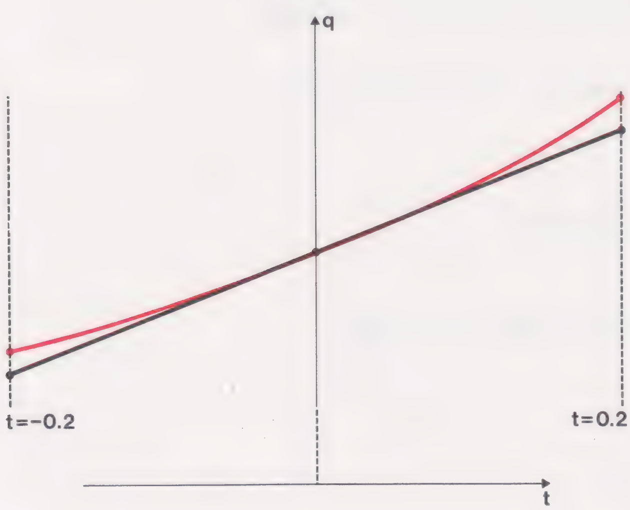


It appears that the graph of V_1 is better than the graph of V as an approximation to the actual graph. ■

Solution 2

Solution 2

Use $Q'(0)$ and march "backwards" until $t = -0.2$. $V(-0.4)$ and so on may be determined in a similar way.



e.g.

$$V(-0.2) = V(0) - 0.2Q'(0)$$
$$= 0.2 - (0.2) \times (0.4)$$
$$= 0.12$$

■

(*continued from page 31*)

If V is the estimated solution we have

$$V(t_0) = q_0$$

$$V(t_1) = q_0 + hQ'(t_0)$$

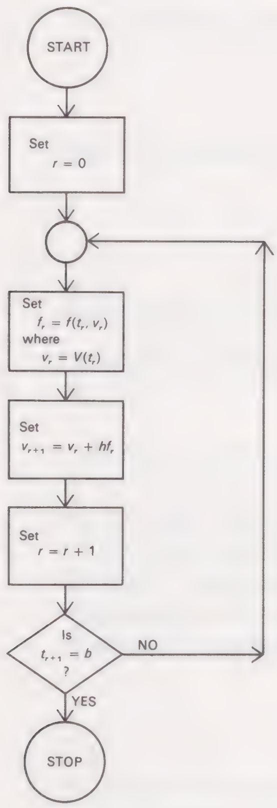
$$= q_0 + hf(t_0, q_0)$$

Then

$$V(t_2) = V(t_1) + hQ'(t_1)$$

$$= v_1 + hf(t_1, v_1), \text{ where } v_1 = V(t_1)$$

and so on. We obtain successive values by repeating this process and this can be represented in the following flow diagram.



This determines a set of $(n + 1)$ points on the approximate solution curve. Notice that this method corresponds to using

$$\frac{\Delta_h}{h} g \text{ as an approximation to } Dg.$$

This can be seen as follows: if

$$DQ(t) = f(t, q)$$

is replaced by

$$\frac{\Delta_h}{h} V(t_r) = f(t_r, v_r),$$

we get

$$\frac{V(t_r + h) - V(t_r)}{h} = f(t_r, v_r),$$

that is

$$V(t_{r+1}) = V(t_r) + hf(t_r, v_r),$$

or

$$v_{r+1} = v_r + hf_r,$$

where

$$f_r = f(t_r, v_r), \qquad v_r = V(t_r).$$

Exercise 3

Exercise 3
(5 minutes)

The differential equation we used to illustrate Euler's method was

$$Q'(t) = 2Q(t)$$

with the initial condition $Q(0) = 0.2$.

For this equation, show that if the step length is h and we start from $t = 0$, then

$$v_r = 0.2(1 + 2h)^r$$

Suppose we are determining the solution curve in the interval [0, 1] and using n lengths; show then that

$$v_n = V(1) = 0.2\left(1 + \frac{2}{n}\right)^n$$

What is the exact value of $Q(1)$? What is $\lim\limits_{n \text{ large}} v_n$? What do you conclude about the sequence of estimated solution curves produced as n increases?

■

Discussion
* *

The last exercise brings us to the problem of the accuracy of Euler's method. Any detailed discussion of this would take us too far: it is technically very complicated, although we already have some of the necessary concepts, such as error interval, error bounds and the correction terms in the Taylor approximations (*Unit 14, Sequences and Limits II*). If you are interested, see the book by Macon listed in the bibliography. But, be warned, it goes well beyond anything we have done in this course, and will prove very difficult at this stage.

Summary

Summary
* *

We have seen in this section an outline of one of the most elementary numerical methods of solving first order differential equations. There are many more advanced methods which increase the accuracy of the estimated solution curve. For example, the Euler method *predicts* the next point from information at the previous point. This new knowledge (the co-ordinates of the next point) enables us to go back and *correct* our estimate of the slope to use over the previous step length. This is just like an iterative process — a first guess, followed by an improvement. Such methods are called *predictor — corrector methods* and lead to important improvements in accuracy.

24.3 CONCLUSION

24.3

Conclusion

*

When introducing differential equations we are always faced with the problem of the mode of presentation. Should we spend a considerable amount of space and time demonstrating how the *mathematical modelling* of a wide variety of problems (such as absorption of drugs into the system, aircraft flight paths or water seepage) produces differential equations, or should we concentrate on the significance of the *solutions* of the differential equations?

We have tried to compromise by selecting a single problem, the population growth problem. We set up the appropriate differential equations and then used this problem as a framework in which to discuss the graphical method, two formula methods and a numerical method of solution. The graphical method enables us to see how solution curves behave, but it only rarely has a direct practical use. The formula methods were fundamental for a long time to finding any solutions at all (and still are to a considerable extent at an elementary level). Numerical methods in general, that is, methods based essentially on the ideas in section 24.2.4, are very powerful methods because of their wide application; however, they always have associated with them the problem of determining the accuracy involved.

Now for a look ahead. There will be another unit on differential equations in which we shall look at an example of a "second order" differential equation; that is, a differential equation involving the second derivative of the desired function. There we shall model the specific problem of vibration.

Acknowledgements

Grateful acknowledgement is made to the following sources for illustrations used in this correspondence text:

Barnabys Picture Library for the photographs which appear on pages 3 and 4; The Mansell Collection for the portrait of Leonhard Euler.

Solution 24.2.4.3

$$v_r = v_{r-1} + hf(t_{r-1}, v_{r-1})$$
$$= v_{r-1} + h \times 2v_{r-1}$$

since, in this case,

$$f(t_{r-1}, v_{r-1}) = 2v_{r-1}$$

Therefore

$$v_r = (1 + 2h)v_{r-1},$$

and by repeated substitution we get

$$v_r = (1 + 2h)^r v_0 = 0.2(1 + 2h)^r$$

For the interval [0, 1],

$$h = \frac{1}{n},$$

so that

$$V(1) = v_n = 0.2\left(1 + \frac{2}{n}\right)^n$$

Now for the actual solution Q,

$$Q(1) = 0.2\, e^2$$

and

$$\lim_{n \text{ large}} v_n = \lim_{n \text{ large}} 0.2\left(1 + \frac{2}{n}\right)^n$$

which, from *Unit 7, Sequences and Limits I*, is also $0.2\, e^2$. The two values are thus the same and we conclude that the limit of the sequence of estimated solution curves coincides with the actual solution curve at both its ends if n is sufficiently large. This suggests that we can get as close to the solution curve as we please in the interval [0, 1], although we have only checked the end-points. ■

Unit No.	Title of Text
1	Functions
2	Errors and Accuracy
3	Operations and Morphisms
4	Finite Differences
5 NO TEXT	
6	Inequalities
7	Sequences and Limits I
8	Computing I
9	Integration I
10 NO TEXT	
11	Logic I — Boolean Algebra
12	Differentiation I
13	Integration II
14	Sequences and Limits II
15	Differentiation II
16	Probability and Statistics I
17	Logic II — Proof
18	Probability and Statistics II
19	Relations
20	Computing II
21	Probability and Statistics III
22	Linear Algebra I
23	Linear Algebra II
24	Differential Equations I
25 NO TEXT	
26	Linear Algebra III
27	Complex Numbers I
28	Linear Algebra IV
29	Complex Numbers II
30	Groups I
31	Differential Equations II
32 NO TEXT	
33	Groups II
34	Number Systems
35	Topology
36	Mathematical Structures